ENUMERATION OF THE FACETS OF CUT POLYTOPES OVER SOME HIGHLY SYMMETRIC GRAPHS

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ABSTRACT. We report here a computation giving the complete list of facets for the cut polytopes over several very symmetric graphs with $15-30$ edges, including K_8, $K_{3,3,3}$, $K_{1,4,4}$, $K_{5,5}$, some other $K_{l,m}$, $K_{1,l,m}$, $Prism_7$, $APrism_6$, Möbius ladder M_{14}, Dodecahedron, Heawood and Petersen graphs.

For K_8, it shows that the huge lists of facets of the cut polytope CUTP_8 and cut cone CUT_8, given in [11] is complete. We also confirm the conjecture that any facet of CUTP_8 is adjacent to a triangle facet.

The lists of facets for $K_{1,l,m}$ with $(l,m) = (4,4), (3,5), (3,4)$ solve problems (see, for example, [29]) in quantum information theory.

1. Introduction

Polyhedra, generated by cuts and by finite metrics are central objects of Discrete Mathematics; see, say, [17]. In particular, they are tightly connected with the well-known NP-hard optimization problems such as the max-cut problem and the unconstrained quadratic $0,1$ programming problem. To find their (mostly unknown) facets is one approach to this problem. It is a difficult problem since the number of facets grows very fast with the size of the problem. However, when full computation is unfeasible, having a partial list of facets can help branch-and-bound strategies that are then commonly used. Here we consider the case of cut polytope over graphs.

Given a graph $G = (V, E)$ with $n = |V|$, for a vertex subset $S \subseteq V = \{1, \dots, n\}$, the *cut semimetric* $\delta_S(G)$ is a vector (actually, a symmetric $\{0,1\}$-matrix) defined as

$$\delta_S(x,y) = \begin{cases} 1 & \text{if } xy \in E \text{ and } |S \cap \{x,y\}| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

So, δ_S can be seen also as the adjacency matrix of a *cut* (into S and $\overline{S}$) *subgraph* of G. Clearly, $\delta_{\{1,\dots,n\}-S} = \delta_S$. If G is connected, which will be the case in our work, there are exactly 2^{n-1} distinct cut semimetrics. The *cut polytope* $\mathrm{CUTP}(G)$ and the *cut cone* $\mathrm{CUT}(G)$ are defined as the convex hull of all such semimetrics and the positive span of all non-zero ones among them, respectively. Their number of vertices, respectively extreme rays is 2^{n-1}, respectively $2^{n-1}-1$ and their dimension is $|E|$, i.e., the number of edges of G.

The most interesting and complicated case is $\mathrm{CUTP}(K_n)$ and $\mathrm{CUT}(K_n)$, denoted simply CUTP_n, CUT_n and called *the cut polytope* and *the cut cone*. In fact, CUT_n is the set of all n-vertex semimetrics, which embed isometrically into some metric space l_1, and rational-valued elements of CUT_n correspond exactly to the n-vertex semimetrics, which embed isometrically, *up to a scale* $\lambda \in \mathbb{N}$, into the path metric of some N-cube K_2^N. It shows importance of this cone in Analysis and Combinatorics.

The *metric cone* MET_n is the set of all *semimetrics on n points*, i.e., those of above functions, which satisfy all *triangle inequalities*, i.e., $d_{ij} \le d_{ik} + d_{kj}$. The bounding of MET_n by $\binom{n}{3}$ *perimeter inequalities* $d_{ij} + d_{ik} + d_{jk} \le 2$ produces the *metric polytope* METP_n. We have the evident inclusions $\mathrm{CUT}_n \subseteq \mathrm{MET}_n$ and $\mathrm{CUTP}_n \subseteq \mathrm{METP}_n$ with $\mathrm{CUT}_n = \mathrm{MET}_n$ and $\mathrm{CUTP}_n = \mathrm{METP}_n$ only for $3 \le n \le 4$. Another relaxation of the cut-polytope is the *hypermetric polytope* considered in [16]. In Table 1 we give the number of facets, vertices (or extreme rays) and the number of orbits for CUTP_n, METP_n with $n \le 8$ and for the corresponding cones.

The symmetry group of a graph $G = (V, E)$ induces symmetry of $\mathrm{CUTP}(G)$. For any $U \subset \{1, \dots, n\}$, the map $\delta_S \mapsto \delta_{U\Delta S}$ also defines a symmetry of $\mathrm{CUTP}(G)$. Together those form the *restricted symmetry group* $ARes(\mathrm{CUTP}(G))$, the order of which is $2^{|V|-1}|\operatorname{Aut}(G)|$. The full symmetry group $\operatorname{Aut}(\mathrm{CUTP}(G))$ may be larger. For $n \ne 4$, $\operatorname{Aut}(\mathrm{CUTP}(K_n)) = ARes(\mathrm{CUTP}(K_n))$ but $\operatorname{Aut}(\mathrm{CUTP}(K_4)) = \operatorname{Aut}(K_{4,4})$ ([15]). So, $|\operatorname{Aut}(\mathrm{CUTP}(K_n))|$ is $2^{n-1}n!$ for $n \ne 4$ and $6 \times 2^3 4!$ for $n = 4$. Let $A(G)$ denote $2^{1-|V|}|\operatorname{Aut}(\mathrm{CUT}(G))|$. One can check the following.

(1) If G is a complete multipartite graph with $t_i \ge 1$ parts of size a_i for $1 \le i \le r$ and $1 \le a_1 < a_2 < \cdots < a_r$, then $|Aut(G)| = \prod_{i=1}^{r} t_i!(a_i!)^{t_i}$.
(2) If G is $Prism_m$, $APrism_m$ or Möbius ladder M_{2m}, then $A(G) = |Aut(G)|$ and, moreover, $A(G) = 4m = 2|V|$ if $m \ne 4$, $m \ge 4$ and $m \ge 4$, respectively.
(3) Examples of infinite series of *singular*, i.e., having $|Aut(\mathrm{CUTP}(G))| > |ARes(\mathrm{CUTP}(G))|$, graphs G are $K_{2,m}$, $K_{1,1,m}$ for any $m \ge 2$ and P_m, C_m for any $m \ge 3$. All 4-, 5- and 6-vertex connected graphs are singular, except $K_{1,3}$, $K_{1,4}$, K_5, $K_5 - e$, $K_5 - e_1 - e_2$ with disjoint edges e_1, e_2 and 19 (among all 112) 6-vertex graphs.

In Table 2 we list information on the cut polytopes of several graphs. Full data sets are available from [18]. The computational techniques used are explained in Section 2; they can be applied to any polytope having a large symmetry group. In Section 3 the results for CUTP_8 are detailed. In Section 4 the results for the correlation polytopes $K_{n,m}$ are presented. Finally, in Section 5 we explain the results for graphs without K_5 minors.

2. Computational methods

The second author has developed over the years an effective computer program ([19]) for enumerating facets of polytopes which are symmetric. The technique used is *adjacency decomposition method*, originally introduced in [11] and applied to the Transporting Salesman polytope, the Linear Ordering polytope and the cut polytope. The algorithm is detailed in Algorithm 1 and surveyed in [10].

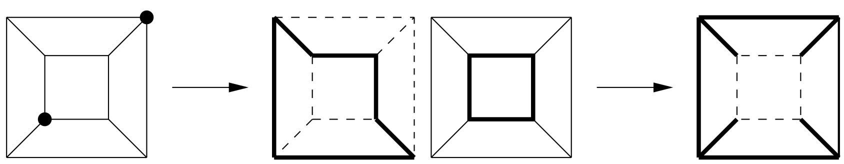
The initial facet of the polytope P is obtained via linear programming. The tests of equivalence are done via the `GAP` functionality of permutation group and their implementation of partition backtrack. The problematic aspect is computing the facets adjacent to a facet. This is itself a dual description problem for the polytope defined by the facet F.

An interesting problem is the check when $\mathcal{R}$ is complete. Of course, if all the orbits are treated, i.e., if $\mathcal{R} = \mathcal{D}$, then the computation is complete. However, sometimes we can conclude before that:

Theorem 1. *Let $G(P)$ be the skeleton graph of a m-dimensional polytope.*

Table 1. The number of facets and vertices (or extreme rays) in the cut and metric polytopes (or cones) for $n \leq 8$. The enumeration of orbits of facets of CUT_n and CUTP_n for $n \leq 7$ was done in [28, 5, 24] for $n = 5$, 6 and 7, respectively. The enumeration of orbits of extreme rays of MET_7 was done in [25]. The orbits of vertices of METP_n were enumerated in [13] for $n = 7$ and in [14] for $n = 8$. For $n \leq 6$ such enumeration is easy.

P	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
CUTP_n, e	$4(1)$	$8(1)$	$16(1)$	$32(1)$	$64(1)$	$128(1)$
$\mathbf{CUTP_n, f}$	$4(1)$	$16(1)$	$56(2)$	$368(3)$	$116,764(11)$	$\mathbf{217,093,472(147)}$
CUT_n, e	$3(1)$	$7(2)$	$15(2)$	$31(3)$	$63(3)$	$127(4)$
$\mathbf{CUT_n, f}$	$3(1)$	$12(1)$	$40(2)$	$210(4)$	$38,780(36)$	$\mathbf{49,604,520(2,169)}$
MET_n, e	$3(1)$	$7(2)$	$25(3)$	$296(7)$	$55,226(46)$	$119,269,588(3,918)$
MET_n, f	$3(1)$	$12(1)$	$30(1)$	$60(1)$	$105(1)$	$168(1)$
METP_n, e	$4(1)$	$8(1)$	$32(2)$	$554(3)$	$275,840(13)$	$1,550,825,600(533)$
METP_n, f	$4(1)$	$16(1)$	$40(1)$	$80(1)$	$140(1)$	$224(1)$



a) A cube and the remaining connected graph obtained from Theorem 1.(i)

b) A cube and the remaining connected graph obtained from Theorem 1.(ii)

Figure 1. Illustration of connectivity results of Theorem 1.

(i) ([6]) $G(P)$ is at least m-connected.

(ii) If we remove all the edges contained in a given face F, then the remaining graph is still connected.

Hence, if the total set of facets, equivalent to a facet in $\mathcal{R} \setminus \mathcal{D}$, has size at most $m-1$ or contains a common vertex, then we can conclude that $\mathcal{R}$ is complete. The requirements for applying any of those two criterions are rather severe, but they are easy to check and, when applicable, the benefits are large. The geometry underlying Theorem 1 is illustrated in Figure 1.

The adjacency decomposition method works well when the polytope is symmetric but it still relies on computing a dual description. This is easy when the incidence of the facet is low but become more and more problematic for facets of large incidence. However, quite often high incidence facets also have large symmetry groups. Therefore, a natural extension of the technique is to apply the method recursively and so, obtain the *recursive adjacency decomposition method*, which is again surveyed in [10].

In order to work correctly, the method requires several ingredients. One is the ability to compute easily automorphism group of the polytope, see [9] for details.

TABLE 2. The number of facets of the cut polytopes CUTP(G) of some graphs. In 4-th column, $A(G)$ denote $2^{1-|V|}|\operatorname{Aut}(\mathrm{CUTP}(G))|$. The case $A(G) > |\operatorname{Aut}(G)|$ is indicated by *

$G=(V,E)$	$\|V\|$	$\|E\|$	$A(G)$	Number of facets (orbits)
K_8	8	28	8!	$\mathbf{217,093,472(147)}$
$K_{3,3,3}$	9	27	$(3)^4$	$\mathbf{624,406,788(2015)}$
$K_{1,4,4}$	9	24	$2(4!)^2$	$\mathbf{36,391,264(175)}$
$K_{1,3,5}$	9	23	3!5!	$\mathbf{71,340(7)}$
$K_{1,3,4}$	8	19	3!4!	$\mathbf{12,480(6)}$
$K_{1,3,3}$	7	15	$2(3!)^2$	$\mathbf{684(3)}$
$K_{1,1,3,3}$	8	21	$4(3!)^2$	$\mathbf{432,552(50)}$
$K_{1,2,m}$, $m \geq 2$	$m+3$	$3m+2$	$\|\operatorname{Aut}(K_{1,2,m})\|$	$\mathbf{8m+8\binom{m}{2}(2)}$
$K_{5,5}$	10	25	$2(5)^2$	$\mathbf{16,482,678,610(1,282)}$
$K_{4,7}$	11	28	4!7!	$\mathbf{271,596,584(15)}$
$K_{4,6}$	10	24	4!6!	$\mathbf{23,179,008(12)}$
$K_{4,5}$	9	20	4!5!	$\mathbf{983,560(8)}$
$K_{4,4}$	8	16	$2(4!)^2$	$\mathbf{27,968(4)}$
$K_{3,m}, m \geq 3$	$m+3$	$3m$	$\|\operatorname{Aut}(K_{3,m})\|$	$\mathbf{6m+24\binom{m}{2}(2)}$
$K_{2,m}, m \geq 3$	$m+2$	$2m$	$2^{m-1}m!\|\operatorname{Aut}(K_{2,m})\|$ *	$\mathbf{4m^2(1)}$
$K_{1,m}, m \geq 2$	$m+1$	m	$m!$	$\mathbf{2m(1)}$
$K_{m+2}-K_m, m \geq 2$	$m+2$	$2m+1$	$2^{m-1}m!\|\operatorname{Aut}(K_{1,1,m})\|$ *	$\mathbf{4m(1)}$
$K_{m+3}-K_m, m \geq 2$	$m+3$	$3m+3$	$3!m!$	$\mathbf{4+12m(2)}$
$K_{m+4}-K_m, m \geq 2$	$m+4$	$4m+6$	$4!m!$	$\mathbf{8(8m^2-3m+2)}(4)$
K_8-K_3	8	25	360	$\mathbf{2,685,152(82)}$
K_7-K_2	7	20	240	$\mathbf{31,400(17)}$
Dodecahedron	20	30	120	$\mathbf{23,804(5)}$
Icosahedron	12	30	120	$\mathbf{1,552(4)}$
Cube	8	12	48	$\mathbf{200(3)}$
Cuboctahedron	12	24	48	$\mathbf{1,360(5)}$
Tr. Tetrahedron	12	18	24	$\mathbf{540(4)}$
$APrism_6$	12	24	24	$\mathbf{2,032(5)}$
$Prism_7$	14	21	28	$\mathbf{7,394(6)}$
$Pyr(Prism_5)$	11	25	20	$\mathbf{208,132(22)}$
$Pyr(APrism_4)$	9	24	16	$\mathbf{389,104(17)}$
Möbius ladder M_{14}	14	21	28	$\mathbf{369,506(9)}$
Heawood graph	14	21	336	$\mathbf{5,361,194(9)}$
Petersen graph	10	15	120	$\mathbf{3,614(4)}$

Another is good heuristics for deciding when to apply the method recursively or not. Yet another is a storing system for keeping dual description that may be reused, and this again depends on some heuristics. The method has been applied successfully on numerous problems [20, 21, 22] and here.

The framework, that we have defined above, can be applied, in order to sample facets of a polytope, say, P. A workable idea is to use linear programming and then

Data: Polytope P and a group G
Result: Set $\mathcal{R}$ of all inequivalent representative of facets of P for G
$F \leftarrow$ facet of P.
$\mathcal{R} \leftarrow \{F\}$.
$\mathcal{D} \leftarrow \emptyset$
while $\mathcal{R}$ *is not complete* **do**
  F a facet in $\mathcal{R} \setminus \mathcal{D}$.
  $\mathcal{D} \leftarrow \mathcal{D} \cup \{F\}$.
  $\mathcal{F} \leftarrow$ facets of P adjacent to F.
  for $F \in \mathcal{F}$ **do**
    test $\leftarrow$ true
    if F *is not equivalent to a facet in* $\mathcal{R}$ **then**
      $\mathcal{R} \leftarrow \mathcal{R} \cup \{F\}$.
    end
  end
end

Algorithm 1: The adjacency decomposition method

get some facets of P. But doing so, we overwhelmingly get facets of high incidence, while we may be interested in obtaining facets of low incidence. One can adapt the adjacency decomposition method to do such a sampling. Let us call two facets *equivalent* if their incidence is the same. By doing so, we remove the combinatorial explosion, which is the main difficulty of such dual-description problem. At the end, we get a number of orbits of facets of different incidence, which give an idea of the complexity of the polytope.

3. The facets of CUTP_8

In [11], the adjacency decomposition method was introduced and was applied to the Transporting Salesman polytope, the Linear Ordering polytope and the cut polytope. For the cut polytope CUTP_8, the authors found 147 orbits, consisting of $217,093,472$ facets, but this list was potentially incomplete, since their method did not treat the triangle, pentagonal and 7-gonal inequalities (defined in Section 3) at that time. Therefore, they only prove that the number of orbits is at least 147. The enumeration is used in [3, 4] for work in quantum mechanics.

Sometimes ([17, 1]), this is incorrectly understood and it is reported that the number of orbits is exactly 147 with [11] as a reference. Here we show that Christof-Reinelt's list is complete. We had to treat the 3 remaining orbits of facets since we could not apply Theorem 1. However, we could apply the theorem in deeper levels of the recursive adjacency method and this made the enumeration faster.

Conjecture 1. *([12, 17]) Any facet of* CUTP_n *is adjacent to a triangle inequality facet.*

The conjecture was checked for $n \leq 7$; here we confirm it for $n = 8$.

Looking for a counterexample to this conjecture, we applied our sampling framework to CUTP_n for $n = 10$, 11 and 12. We got initial facets of low incidence and then we complemented this with random walks in the set of all facets. This allowed us to find many simplicial facets (more than $10,000$ for each) of these CUTP_n but all of them were adjacent to at least one triangle inequality facet.

As a corollary to the confirmation of above conjecture for $n = 8$, we obtain that the ridge graphs of CUTP_8 and CUT_8 have diameter 4. In fact, the subgraphs, formed by the triangle facets, have diameter 2.

4. Correlation polytopes of $K_{n,m}$

In quantum physics and quantum information theory, Bell inequalities, involving joint probabilities of two probabilistic events, are exactly inequalities valid for the *correlation polytope* $\mathrm{CORP}(G)$ (called also *Boolean quadric polytope* $BQP(G)$) of a graph, say, G. In particular, $\mathrm{CORP}(K_{n,m})$ is seen in quantum theory as the set of possible results of a series of Bell experiments with a non-entangled (separable) quantum state shared by two distant parties, where one party (Alice) has n choices of possible two-valued measurements and the other party (Bob) has m choices. Here, a valid inequality of $\mathrm{CORP}(K_{n,m})$ is called a *Bell inequality* and if facet inducing, a *tight Bell inequality*. This polytope is equivalent (linearly isomorphic via the *covariance map*) to the cut polytope of $K_{1,n,m}$ [17, Section 5.2]. Similarly, $\mathrm{CUTP}(K_{1,n,m,l})$ represents three-party Bell inequalities.

The symmetry group of $\mathrm{CORP}(K_{n,n})$ and $\mathrm{CUTP}(K_{1,n,m})$ is of order $2^{1+n+m}n!m!$.

We computed the facets of $\mathrm{CORP}(K_{n,m})$ having $(n, m) = (2, m)$ with $m \leq 6$, $(3, 4), (3, 5), (4, 4)$ and confirmed known enumerations for $(n, m) = (2, 2), (3, 3)$. In fact, the page [29] collects the progress on finding Bell inequalities. The case of $\mathrm{CORP}(K_{n,n})$ is called there $(2, n, 2)$-*setting*. The cases $n = 2, 3$ were settled in [23] and [26], respectively. For $n = 4$, partial lists of facets were known; our 175 orbits of $36, 391, 264$ facets of $\mathrm{CORP}(K_{4,4})$ finalize this case.

In contrast to the Bell's inequalities, which probe entanglement between spatially-separated systems, the *Leggett-Garg inequalities* test the correlations of a single system measured at different times. The polytope, defined by those inequalities for n observables, is, actually ([2]), the cut polytope CUTP_n.

5. Cut polytopes of graphs without minor K_5

Given a graph $G = (V, E)$, an edge $(v_1v_2) \in E$ defines an *edge inequality* $x(v_1, v_2) \geq 0$. Similarly, an s-cycle $(v_1, \dots, v_s)$ of G with $s \geq 3$ defines a *cycle inequality*

$$\sum_{i=1}^{s-1} x(v_i, v_{i+1}) - x(v_1, v_s) \geq 0.$$

These inequalities and their switching define valid inequalities on $\mathrm{CUTP}(G)$. There are $2|E|$ edge inequalities and their incidence is $2^{|V|-2}$. Each s-cycle of G gives 2^{s-1} s-cycle faces, which are of incidence $s2^{|V|-s}$.

In fact, [8, 7] and [27], gives, respectively, that:

(i) only chordless s-cycles and edges not containing in a 3-cycle produces facets;

(ii) no other facets exist if and only if G is a K_5-minor-free graph.

All K_5-minor-free graphs in Table 2 are $K_{1,2,m}$; $K_{2,m}$; $K_{m+i} - K_m$ with $m \geq 2, 1 \leq i \leq 3$ and the skeletons of regular and semiregular polyhedra (Dodecahedron, Icosahedron, Cube, Cuboctahedron, truncated Tetrahedron, $APrism_6, Prism_7$).

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